III Classification of supergravity solutions II

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Hand-in problem 2:

Compute the integrability conditions of the Killing spinor equations for 11D SUGRA, using GAMMA and the supplied "getting started" Mathematica notebook, analogous to eq. (2.9) in hep-th/0503046 (Systematics of M-theory spinorial geometry) but use instead the $\Gamma^{(3)}$ contraction, i.e. express $\Gamma^{ABC} R_{BC}$ in terms of the field eqs (2.10).

5 Recap

Spinorial geometry – An efficient method for solving the KSE.
[hep-th/0410155]

Three key ingredients :

* Spinors in terms of forms
* Simplify the spinors using gauge symmetry
* An oscillator basis for Γ-matrices

By solving the KSE we mean

$$\boxed{\text{Killing spinors} \Longrightarrow \text{metric \& fluxes}}$$

This gives a susy geometry / configuration

Some field eqs are automatically satisfied, but generally not all

$$\boxed{\text{SUSY geometry/config} \xrightarrow{\text{field eqs}} \text{SUSY solution}}$$

Spinors in terms of forms

Ex: $\underset{\in \mathbb{C}}{c_1}\,\underbrace{e_1 \wedge e_2 \wedge e_3}_{\text{real basis forms}}$

Different (Weyl) chiralities correspond to odd or even forms. To get Majorana spinors we have to enforce a reality condition.

In the oscillator basis the Γ-matrices act as creation and annihilation ops

Ex: $\begin{cases} \Gamma^1 = \sqrt{2}\, e_1 \lrcorner \\ \Gamma^{\bar{1}} = \sqrt{2}\, e_1 \wedge \end{cases}$

To evaluate expressions like $\Gamma^{\alpha\bar{\beta}} \cdot 1$

we use 1) The Clifford algebra

2) That $\Gamma^{\alpha} \cdot 1$ annihilates 1

For more complicated spinors split the indices into more than one type, e.g. e_{123}

$\Rightarrow$ $\alpha = 1,2,3$ & $p = 4,5$ and use that

$$\Gamma^{\bar{\alpha}} e_{123} = \Gamma^{p} e_{123} = 0$$

6 Comments on complex conjugation

Always complex conjugate using the C operator!

<u>Ex:</u> 4D $\quad C = -\Gamma_{012} * = (e_{1\wedge} + e_{1\lrcorner})(-e_{2\wedge}e_{2\lrcorner} + e_{2\lrcorner}e_{2\wedge}) *$

$$\underline{\underline{C\gamma_+}} = C\sqrt{2}\, e_{2\lrcorner} = \sqrt{2}\,(e_{1\wedge} + e_{1\lrcorner})(-e_{2\wedge}\cancel{e_{2\lrcorner}}e_{2\lrcorner} + e_{2\lrcorner}e_{2\wedge}e_{2\lrcorner}) *$$

$$= -\sqrt{2}\, e_{2\lrcorner}(e_{1\wedge} + e_{1\lrcorner})\, e_{2\wedge}e_{2\lrcorner} *$$

$$= +\sqrt{2}\, e_{2\lrcorner}(e_{1\wedge} + e_{1\lrcorner})(-e_{2\wedge}e_{2\lrcorner} + e_{2\lrcorner}e_{2\wedge}) * = \underline{\underline{\gamma_+ C}}$$

Similarly $\underline{\underline{C\gamma_-}} = \underline{\underline{\gamma_- C}} \quad \Rightarrow \quad \boxed{\gamma_+ \,\&\, \gamma_- \text{ real}}$

$$\underline{\underline{C\gamma_1}} = C\sqrt{2}\, e_{1\wedge} = \sqrt{2}\,(e_{1\wedge} + e_{1\lrcorner})\, e_{1\wedge}(-e_{2\wedge}e_{2\lrcorner} + e_{2\lrcorner}e_{2\wedge}) *$$

$$= +\sqrt{2}\, e_{1\lrcorner}e_{1\wedge}(-e_{2\wedge}e_{2\lrcorner} + e_{2\lrcorner}e_{2\wedge}) *$$

$$= \underbrace{+\sqrt{2}\, e_{1\lrcorner}}_{=\gamma_{\bar{1}}}\underbrace{(e_{1\wedge} + e_{1\lrcorner})(\qquad\qquad) *}_{=C} = \underline{\underline{\gamma_{\bar{1}} C}}$$

Similarly $\underline{\underline{C\gamma_{\bar{1}}}} = \underline{\underline{\gamma_1 C}} \quad \Rightarrow \quad \boxed{\gamma_1 \,\&\, \gamma_{\bar{1}} \text{ complex}}$

<u>NB:</u> $\gamma_1 = \sqrt{2}\, e_{1\wedge}$ naively looks real but it's not!

Covariance implies that the indices on the other fields behave in the same way

<u>Ex:</u> $(\Omega_{1,-+})^* = \Omega_{\bar{1},-+}$

7 G-structures [Grey & Hervella, '80]

One common way of describing geometries (e.g. solutions to the KSEs) is in terms of G-structures.

In 11D we had two $N=1$ orbits:

isotropy / stability sub-group

$$\begin{cases} \epsilon = f(1 + e_{12345}) & G = SU(5) \\ \epsilon = 1 + e_{1234} & G = (Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R} \end{cases}$$

G-structure!

The so called <u>intrinsic torsion classes</u> are simply given by the irreps (under G) of the <u>spin connection</u> excluding $\Omega \in G$.

<u>Ex:</u> $G = SU(5)$ $\quad \alpha = 1, .., 5$

$$\Omega_{A,BC} = \{ \overset{\in SU(5)}{\cancel{\Omega_{\bar\alpha,\beta\bar\gamma}}}, \Omega_{\bar\alpha,\beta\gamma}, \Omega_{\bar\alpha,\bar\beta\bar\gamma}, \Omega_{\bar\alpha,\beta}{}^{\beta} \}$$

$$\Rightarrow \begin{cases} (W_3)_\alpha := \Omega_{\bar\beta,}{}^{\bar\beta}{}_{\alpha} \quad , \quad (W_4)_{\bar\alpha\beta\gamma} = \Omega_{\bar\alpha,\beta\gamma} - \frac{1}{2}\Omega_{\bar\delta,}{}^{\bar\delta}{}_{[\gamma}\, g_{\beta]\bar\alpha} \\ (W_5)_{\bar\alpha} = \Omega_{\bar\alpha,\beta}{}^{\beta} \\ (W_1)_{\overline{\alpha\beta\gamma}} = \Omega_{[\bar\alpha,\overline{\beta\gamma}]} \quad , \quad (W_2)_{\overline{\alpha\beta\gamma}} = \frac{2}{3}\Omega_{\bar\alpha,\overline{\beta\gamma}} - \frac{1}{3}\Omega_{\bar\gamma,\overline{\alpha\beta}} - \frac{1}{3}\Omega_{\bar\beta,\overline{\gamma\alpha}} \end{cases}$$

$(W_i) = 0, \ \forall i \Rightarrow$ <u>special holonomy</u>

(8) Integrability conditions and field equations

When looking for explicit solutions it is very useful to know the minimal set of field equations that need to be solved. This information can be obtained by studying the integrability conditions to the KSE.

Look at 11D SUGRA.

supercovariant connection

KSE: $\mathcal{D}_A \epsilon = \nabla_A \epsilon + \Sigma_A \epsilon$

where
$$\begin{cases} \nabla_A \epsilon = \partial_A \epsilon + \frac{1}{4}\Omega_{A,BC}\Gamma^{BC} & \text{(Levi-Civita connection)} \\ \Sigma_A = -\frac{1}{288}\left(\Gamma_A{}^{B_1\cdots B_4} - 8\,\delta_A^{B_1}\Gamma^{B_2B_3B_4}\right)F_{B_1\cdots B_4} \end{cases}$$

If the KSE is satisfied, $\mathcal{D}_A \epsilon = 0$,

then surely

$$\boxed{[\mathcal{D}_A, \mathcal{D}_B]\epsilon := \mathcal{R}_{AB}\,\epsilon = 0}$$

supercovariant curvature

It is easy to compute $\mathcal{R}_{AB}$, e.g. using GAMMA, but the result is a long and messy expression.

However, from $\mathcal{R}_{AB}$ we can compute

1) $\Gamma^B \mathcal{R}_{AB}$

2) $\Gamma^{ABC} \mathcal{R}_{BC}$ ← Gives a simpler result! (hand-in problem)

and these results can be expressed in terms of the field equations!

Ex: $$I_A \epsilon := \Gamma^B \mathcal{R}_{AB} = \Big[E_{AB}\Gamma^B + L_{C_1C_2C_3}\left(\Gamma_A{}^{C_1C_2C_3} - 6\,\delta_A^{C_1}\Gamma^{C_2C_3}\right) + B_{C_1\cdots C_5}\left(\Gamma_A{}^{C_1\cdots C_5} - 10\,\delta_A^{C_1}\Gamma^{C_2\cdots C_5}\right)\Big]\epsilon = 0$$

where

$$\left\{ \begin{aligned} E_{AB} &:= \underset{\substack{\downarrow \\ \text{ordinary curvature}}}{R_{AB}} - \frac{1}{12} F_{AC_1C_2C_3} F_B{}^{C_1C_2C_3} + \frac{1}{144} g_{AB} F_{C_1\cdots C_4} F^{C_1\cdots C_4} \\ L_{ABC} &:= -\frac{1}{36} * \left(d * F - \frac{1}{2} F \wedge F \right)_{ABC} \\ B_{A_1\cdots A_5} &:= \frac{1}{6!} (dF)_{A_1\cdots A_5} \end{aligned} \right.$$

(11D SUGRA — Gauntlett & Pakis — hep-th/0212008

IIB — Gran et al. — hep-th/0507087)

↑ More complicated since there are two KSE

Note now that the contracted integrability condition I_{AE} is in form very similar to the KSE as it is linear in the "fields" E_{AB}, $L_{C_1 \ldots C_3}$ and $B_{C_1 \ldots C_5}$ (insted of $\Omega_{A,BC}$ and $F_{C_1 \ldots C_4}$)

$\Rrightarrow$ Use spinorial geometry!

$$\boxed{\begin{array}{ccc} \text{Killing spinors} & \Rrightarrow & \text{linear relations between } E_{AB},\ L_{C_1C_2C_3} \text{ and } B_{C_1 \ldots C_5} \end{array}}$$

first order!

One strategy is to express for E_{AB} and $L_{C_1C_2C_3}$ (as much as possible) in terms of $B_{C_1 \ldots C_5}$, which is easy to solve being first order in derivatives.

$\Rrightarrow$ Minimal set of field equation that need to be solved in order to ensure that all field eqs are satisfied.

Considering the difficulty of analysing the field eqs (2nd order non-linear diff. eqs) this is a huge simplification.

(9) Intro to GAMMA [hep-th/0108113]

The package is compatible with Mathematica versions up to V5.2.

(*) Loading the package

```
<< C:\[Path]\GAMMAV20.m
```

(*) Setting the spacetime and spinor dimensions

```
SetDim[11]
SetSpinorDim[32]
```

← Used when computing traces

↑ **NB**: Case sensitive!

(*) Commenting your code

```
(* comment *)
```

(*) Γ-matrices are written using

```
GammaProd[{a1,...,ap}, {b1,...bq}, ... ]
```

represents the product

$$\Gamma^{a_1 \cdots a_p} \Gamma^{b_1 \cdots b_q} \cdots$$

↑ can also enter expressions in this form.

NB: All spacetime indices are written downstairs ⟹ no problem, in a pair of contracted indices any of the two indices can be viewed to be upstairs.

(*) Expanding products of Γ-matrices, i.e. using the Clifford algebra to get a linear expression in $\Gamma^{(p)}$, is done by

GammaExpand[ ]

or GE[ ] ← short version

(*) To simplify expressions contracted indices, so called "dummy indices" may need to be renamed, in e.g. $A_m A_m + A_n A_n$

RenameDummy[ ] or RD[ ]

(*) To anti-symmetrise free (i.e. uncontracted) indices

ASym[expr, {indices}]

↑ list of the indices to anti-symmetrise

To symmetrise

Sym[expr, {indices}]

(*) To represent a tensor, e.g. the 4-form $F_{A_1A_2A_3A_4}$ use

Tensor[F, {A1, A2, A3, A4}]

always treated as anti-symmetric in these indices.

This can also be used to represent the covariant derivative $D_M \equiv \nabla_M$

The notation we have used previously

Tensor[D, {M}]

But note that D_M is then treated as a vector and is moved around in the expression

$\Rightarrow$ keep track of what it hits by hand!

(*) Different expressions with Γ-matrices do not commute and has to be multiplied together using **

NonCommutative Multiply

Note that each index can appear at most <u>twice</u> (i.e. contrated) in the whole expression!

Multiply[expr1, expr2, ...] takes care of this problem by renaming indices if necessary.

(*) To get information on how to use a specific command write

?command

(*) Metrics / Kronecker deltas are written using

Delta[{a1, ..., ap}, {b1, ..., bp}]

representing $\delta^{a_1 \dots a_p}_{[b_1 \dots b_p]}$

(*) To put manifestly antisymmetric indices back into a canonical order, to simplify expressions, use

ACanonicalOrder[expr, {indices}]

or AC[expr, {indices}]

VI Special commands in the supplied notebook

(*) DualiseGammas[expr]

Dualises all $\Gamma^{(p)}$ with $p > 5$

(*) SelectGamma[expr, p]

Picks out all $\Gamma^{(p)}$ terms in expr

$\boxed{\text{See also the comments and definitions in the supplied notebook!}}$

(10) Systematics of spinorial geometry

Suppose you want to use spinorial geometry to analyse, or look for, solutions to 11D, IIA/B or type I/heterotic supergravity (the most difficult cases due to the high dimension, and in IIB the additional complication of having complex spinors)

<u>Step 1</u>: Determine the Killing spinors

- Use the Killing spinors of a known solution in order to look for a generalisation
- Use a classification
- Expected G-structure / physical intuition
- Guess
- Etc...

<u>Step 2</u>: For <u>all</u> basis spinors both the KSE and integrability conditions have been evaluated in

[hep-th/0503046] for 11D SUGRA ($\Rightarrow$ IIA)

[hep-th/0507087] for IIB SUGRA

$\Rightarrow$ Type I / heterotic

This means that for an arbitrary set of Killing spinors the resulting linear systems in terms of the spin connection and fluxes (or the field equations) can be written down, using the results in the papers above, without any computations!

Step 3: Solve the linear system, which is trivial.

$\because$ Solving the KSE or the related integrability conditions has been reduced to just solving a linear system!

This simplification makes the analysis of a large number of cases feasible, e.g. classifying various types of solutions. The main problem in a classification is reduced to finding the different sets of Killing spinors (using gauge symmetry is crucial in order to reduce the number of cases).