Assignment 1

Problem

Consider $N = (1,1)$ sigma model written in $N = (1,1)$ superfields

$$S = \int d^2\sigma\, d^2\theta\ D_+\Phi^\mu D_-\Phi^\nu g_{\mu\nu}(\Phi)\ , \tag{1}$$

where g is metric tensor. The action (1) is manifestly supersymmetric under the usual supersymmetry transformations

$$\delta_1(\epsilon)\Phi^\mu = -i(\epsilon^+ Q_+ + \epsilon^- Q_-)\Phi^\mu\ , \tag{2}$$

which form the standard supersymmetry algebra

$$[\delta_1(\epsilon_1), \delta_1(\epsilon_2)]\Phi^\mu = -2i\epsilon_1^+\epsilon_2^+\partial_{+\!\!+}\Phi^\mu - 2i\epsilon_1^-\epsilon_2^-\partial_=\Phi^\mu\ . \tag{3}$$

We may look for additional supersymmetry transformations of the form

$$\delta_2(\epsilon)\Phi^\mu = \epsilon^+ D_+\Phi^\nu J^\mu_\nu(\Phi) + \epsilon^- D_-\Phi^\nu J^\mu_\nu(\Phi)\ . \tag{4}$$

Classically the ansatz (4) is unique for dimensional reasons.

Please, show that the transformations (4) obey the same algebra as (3) if J is a complex structure. Moreover the action (1) is invariant under the transformations (4) if the manifold is Kähler.

The first supersymmetry transformations (2) and the second supersymmetry transformations (4) automatically commute

$$[\delta_2(\epsilon_1), \delta_1(\epsilon_2)]\Phi^\mu = 0\ . \tag{5}$$

Notation

Here we collect the notation for $N = (1,1)$ superspace.

We use real (Majorana) two-component spinors $\psi^\alpha = (\psi^+, \psi^-)$. Spinor indices are raised and lowered with the second-rank antisymmetric symbol $C_{\alpha\beta}$, which defines the spinor inner product:

$$C_{\alpha\beta} = -C_{\beta\alpha} = -C^{\alpha\beta}\ , \qquad C_{+-} = i\ , \qquad \psi_\alpha = \psi^\beta C_{\beta\alpha}\ , \qquad \psi^\alpha = C^{\alpha\beta}\psi_\beta\ . \tag{6}$$

Throughout the paper we use $(+\!\!+, =)$ as worldsheet indices, and $(+,-)$ as two-dimensional spinor indices. We also use superspace conventions where the pair of spinor coordinates of the two-dimensional superspace are labelled $\theta^{\pm}$, and the spinor derivatives $D_{\pm}$ and supersymmetry generators $Q_{\pm}$ satisfy

$$\begin{aligned} D_+^2 &= i\partial_{\mathbin{+\!\!\!+}} \ , \qquad D_-^2 = i\partial_{=} \ , \qquad \{D_+, D_-\} = 0 \ , \\ Q_{\pm} &= iD_{\pm} + 2\theta^{\pm}\partial_{\substack{+\!\!\!+\\ =}} \ , \end{aligned} \tag{7}$$

where $\partial_{\substack{+\!\!\!+\\ =}} = \partial_0 \pm \partial_1$. The supersymmetry transformation of a superfield Φ is given by

$$\begin{aligned} \delta\Phi &\equiv -i(\varepsilon^+ Q_+ + \varepsilon^- Q_-)\Phi \\ &= (\varepsilon^+ D_+ + \varepsilon^- D_-)\Phi - 2i(\varepsilon^+\theta^+\partial_{\mathbin{+\!\!\!+}} + \varepsilon^-\theta^-\partial_{=})\Phi \ . \end{aligned} \tag{8}$$

The components of a scalar superfield Φ are defined by projection as follows:

$$\Phi| \equiv X \ , \qquad D_{\pm}\Phi| \equiv \psi_{\pm} \ , \qquad D_+D_-\Phi| \equiv F \ , \tag{9}$$

where the vertical bar $|$ denotes "the $\theta = 0$ part". The $N = (1,1)$ spinorial measure is conveniently written in terms of spinor derivatives:

$$\int d^2\theta \ \mathcal{L} = (D_+D_-\mathcal{L})\Big| \ . \tag{10}$$