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Faraday Rotation and GPS

Take-home Quiz
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Abstract

The Faraday rotation effect is reviewed, and the effect is estimated for GPS frequencies and a typical ionosphere. It is noted that GPS satellites do not use linearly polarized transmissions, so in reality, there is no Faraday rotation effect on GPS signals. The measurement of electron columnar count with other techniques is discussed, and it is found that ionospheric group delay can be used for finding the total electron content (TEC) with GPS.

1 Introduction: The Ionosphere

In 1901, Marconi received a radio signal across the Atlantic, despite the curvature of the Earth. Soon it was realized that a conducting layer at high altitude caused the effect.[1] The existence of the ionosphere was finally proven experimentally in 1925 [1].

Thus the ionosphere, a shell of weakly ionized plasma, was discovered through radio broadcasting. The physics of the ionosphere is of even greater interest in radio science today, with the advent of large-scale radio communication. Much is available about the ionosphere in the literature [2, 9], so here only the bare concepts needed for the present problem will be reviewed.

The ionization which earns this part of the atmosphere its name is caused primarily by ultraviolet radiation from the Sun; the species involved in the reactions vary with the abundance of that particular species at different heights. However, in addition to photodissociation and recombination processes, there are also diffusion effects and combination by collisions complicating a complete description of the formation of the ionosphere. A reasonably successful theory was initiated in the 1930s by Chapman (for a review, consult [2]), in which the reactions are confined to independent layers of the ionosphere. Today, the layers of the ionosphere are called D, E, F1 and F2, named in order of increasing height.

The extent of the ionosphere depends on one's interests. For radio propagation this will be determined by the frequency. For very low frequencies the ionosphere begins at 50 km height (D layer), but for the scope of this project (MHz frequencies and higher), the main regions are the F1 layer (140-210 km) [4], the F2 layer (210-1,000 km) and to some extent the protosphere (1000 km up to satellite distances).

The electron density in the ionosphere is sketched in Fig. 1. One should also note that while the maximum density lies somewhere in the 300-350km region, the ratio of the total electron content[2] to the electron content below the maximum has been measured to lie around 4:1 (Fig. 5). Thus, most of the distribution lies above the maximum.

2 Faraday Rotation

A linearly polarized wave propagating along a magnetic field in a plasma will suffer a rotation of its plane of polarization, called a Faraday rotation, after M. Faraday, who found the effect in lead borate glass in the year 1845 [3].

Mathematically, the effect arises when one solves the field equations and sets the determinant of the field coefficient matrix equal to zero; the resulting equation has two roots, making for a birefringence phenomenon: [6, 7, 8]

$$n_{\pm}^2 = 1 - \frac{\omega_p^2/\omega^2}{1 \pm \omega_c/\omega}$$

where ω_p is the plasma frequency and ω_c the cyclotron frequency.

One can show [8] that the two different waves are right or "R" (for the minus) and left or "L" (for the plus) circularly polarized. Physically, the presence of a longitudinal magnetic field can be seen to either increase or reduce the curvature of the electron trajectory depending on the sense of polarization, and therefore increase or decrease the amplitude of electron motion. This, in turn, diminishes or enhances the dipole character of the electrons, and for large ω, the different dielectric constants mean the R wave travels faster than the L wave.

[1] Kennelly and Heaviside in 1902. This region was, consequently, called the Kennelly-Heaviside region but now goes under the name of "normal E region". [9]

[2] in a vertical column extending from the observer upwards, measured in electrons/m^2

Indeed, for $\omega \gg \omega_c, \omega_p$, we find

$$\begin{aligned} n_\pm &= \left(1 - \frac{\omega_p^2/\omega^2}{1 \pm \omega_c/\omega}\right)^{1/2} \\ &\simeq 1 - \frac{1}{2}\,\frac{\omega_p^2/\omega^2}{1 \pm \omega_c/\omega} + \mathcal{O}\left(\frac{\omega_p^4}{\omega^4}\right) \\ &\simeq 1 - \frac{\omega_p^2}{2\omega^2}\left(1 \mp \omega_c/\omega + \mathcal{O}(\omega_c^2/\omega^2)\right) + \mathcal{O}\left(\omega_p^4/\omega^4\right) \end{aligned} \tag{1}$$

A linearly polarized wave can be composed of two circularly polarized waves. This is clear if one chooses suitable polarization vectors: The linearly polarized wave (with the plane of polarization at an angle θ_0 to the vertical) is $\mathbf{E} = (E\cos\theta_0\,\hat{\epsilon}_x + E\sin\theta_0\,\hat{\epsilon}_y)\,e^{-i\omega t}$. Rewrite $\mathbf{E}$ in a new basis:

$$\hat{\epsilon}_\pm = \frac{1}{\sqrt{2}}\,(\hat{\epsilon}_x \pm i\hat{\epsilon}_y) \tag{2}$$

which gives

$$\begin{aligned} \mathbf{E} &= \frac{1}{\sqrt{2}}\,[(E\cos\theta_0 - iE\sin\theta_0)\hat{\epsilon}_+ + (E\cos\theta_0 + iE\sin\theta_0)\,\hat{\epsilon}_-]\,e^{-i\omega t} \\ &= \frac{E}{\sqrt{2}}\,[e^{-i\theta_0}\,\hat{\epsilon}_+ + e^{i\theta_0}\,\hat{\epsilon}_-]\,e^{-i\omega t} \end{aligned} \tag{3}$$

For circularly polarized light, we simply choose the $\hat{\epsilon}_-$ component (L wave) or $\hat{\epsilon}_+$ component (R wave)[3] . Change the notation accordingly: $\hat{\epsilon}_{+,-} \to \hat{\epsilon}_{R,L}$.

2.1 The Effect

Let the linearly polarized wave (3) propagate through the plasma; since R travels faster (for high frequencies) it will have gone through fewer cycles upon arrival. If the angles are θ_L and θ_R, we have

$$\mathbf{E} = \frac{E}{\sqrt{2}}\,[e^{i\theta_R}\,e^{-i\theta_0}\,\hat{\epsilon}_R + e^{i\theta_L}\,e^{i\theta_0}\,\hat{\epsilon}_L]\,e^{-i\omega t}$$

With new variables

$$\phi = \frac{\theta_L + \theta_R}{2} \qquad \theta = \frac{\theta_L - \theta_R}{2}$$

we find

$$\begin{aligned} \mathbf{E} &= \frac{E}{\sqrt{2}}\,[e^{i(\phi-\theta)}e^{-i\theta_0}\hat{\epsilon}_R + e^{i(\phi+\theta)}e^{i\theta_0}\,\hat{\epsilon}_L]\,e^{-i\omega t} \\ &= \frac{E}{\sqrt{2}}\,[e^{-i(\theta_0+\theta)}\,\hat{\epsilon}_R + e^{i(\theta_0+\theta)}\,\hat{\epsilon}_L]\,e^{-i(\omega t-\phi)} \end{aligned}$$

which paid off: now the angle of the plane of polarization is $\theta_0 + \theta$ and the phase shift is ϕ. Therefore, the Faraday rotation angle after passing through a plasma of length dz is θ, where

$$\begin{aligned} \theta &= \frac{\theta_L - \theta_R}{2} = \frac{k_L\,dz - k_R\,dz}{2} = \frac{\omega}{2c}\,(n_L - n_R)\,dz \\ &\overset{eqn.(1)}{=} \frac{\omega}{2c}\,\frac{\omega_p^2}{2\omega^2}\,[1 + \omega_c/\omega - (1 - \omega_c/\omega)]\,dz = \frac{\omega_p^2\,\omega_c}{2\,c\,\omega^2}\,dz \end{aligned}$$

[3]At this stage, it is not obvious which wave actually corresponds to which sign, but it can be shown in a straightforward manner [8] which will not be repeated here.

so in SI units, for propagation along a magnetic field line, the Faraday rotation angle is

$$\begin{aligned}\theta &= \frac{e^3}{8\pi^2 c\,\epsilon_0\, m_e^2}\,\frac{1}{f^2}\int_{\text{ionosphere}} B(z)n(z)dz \\ &\simeq \frac{1.35\cdot 10^6}{f^2}\int B(z)n(z)dz \ \ \text{(degrees)} \\ &\simeq \frac{1.35\cdot 10^6}{f^2}\,\bar{B}\int n(z)dz \ \ \text{(degrees)} \qquad (4)\end{aligned}$$

where in the last step the magnetic field was observed to be largely constant throughout the ionosphere and was replaced by its value $\bar{B}$ at the peak electron density height. The integral is now the total electron count (TEC). Above the Austin (Texas) area, typical TEC range from $1.8\cdot 10^{17}$ to $6.5\cdot 10^{17}$ (see Fig. 2). The magnetic field $\bar{B}$ at the peak height 300-350 km is around $3\cdot 10^{-5}$ T [10], so an estimate for the *difference* in Faraday rotation for two *linearly* polarized frequencies $f_1 = 1575$ MHz and $f_2 = 1227$ MHz would be

$$\theta_{2-1} = 1.35\cdot 10^6\left(\frac{1}{(1.227\cdot 10^9)^2} - \frac{1}{(1.575\cdot 10^9)^2}\right) 3\cdot 10^{-5}\times[1.8, 6.5]\times 10^{17} = [1.9^\circ, 6.9^\circ]$$

where the square brackets denote an interval. If the signal is received on a dipole antenna, the rotation of the plane results in a potentially measurable, if small, periodic fading. The differential character of this method helps remove systematic errors in the rotation angle.

To summarize, we see that the Faraday rotation of linearly polarized electromagnetic waves is fairly small at GHz frequencies, and it falls of as $1/f^2$.

2.2 Corrections

This calculation is only to first order, and so there is little incentive for patching up a crude model by small corrections. However, one correction that might be of importance is for the oblique incoming angle. We have assumed the signal is going through the ionosphere along a magnetic field line, thus the satellite cannot be straight up ("elevation angle 90°") but is at an angle equal to that of the inclination of the Earth's magnetic field in the ionosphere. The reason this matters for the Faraday rotation is that a greater distance in the ionosphere will be traversed if the signal is coming in at an oblique angle ("elevation angle $< 90^\circ$") and the TEC is defined as the content in a *vertical* column. For elevation angles of e.g 70°, the obliquity factor Q (ratio of traversed TEC to vertical TEC) is around 1.2, so the Faraday rotation angle would be increased by 20% [11]. The obliquity factor Q goes up to 3 for a 0° inclination, however, that will not be the case.

Also, see section 4.3.

3 GPS

The GPS (Global Positioning System) satellite system is primarily designed for pinpointing the position of an observer down to previously unheard-of precision. The system is based on 24 geostationary satellites orbiting at an altitude of 20,162.61 km [11].

For satellite studies of the atmosphere, GPS is a gold mine. Radio studies have been done by moon echoing, rockets, non-stationary satellites at lower orbits (a few thousand kilometers) and other geostationary systems than GPS, but the GPS satellites provide ionosphere researchers with an omnipresent, reliable source of additional information.

However, the GPS signals are transmitted with right-hand circular polarization [4]. Clearly, there can be no Faraday rotation[4] of a circularly polarized signal. In fact,

$$
\begin{aligned}
\mathbf{E} &= \frac{E}{\sqrt{2}} e^{i\theta_R} \, e^{-i\theta_0} \hat{\epsilon}_R \, e^{-i\omega t} \\
&= \frac{E}{\sqrt{2}} e^{i\theta_R} \, e^{-i(\omega t+\theta_0)} \, \hat{\epsilon}_R
\end{aligned}
$$

where θ_R is now a shift compared to the phase which would have been received in the absence of an ionosphere. Thus, if one wants to measure the TEC with GPS signals, one must resort to other methods.

4 Other Techniques For TEC Measurement

4.1 Phase Shift

Can the above-mentioned phase shift be used? The GPS frequencies L_1 and L_2 are phase-locked [11] so one can consider how many wavelengths (cycles) $d\varphi$ are traversed during passage through an ionosphere thickness dl with the new wavelength $\lambda_n = \lambda/n$:

$$
d\varphi = \frac{dl}{\lambda_n} = \frac{n}{\lambda} \, dl
$$

and the change in cycles over a distance L is

$$
\begin{aligned}
d\phi &= \frac{L}{\lambda} - d\varphi \\
\Delta\phi &= \int d\phi = \frac{1}{\lambda} \int (1-n) dl
\end{aligned}
$$

This is called the ionospheric *carrier phase advance*. To lowest order,

$$
n = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} \simeq 1 - \frac{\omega_p^2}{2\omega^2}
$$

Thus, in SI units.

$$
\Delta\phi = \frac{f}{2c} \int \frac{ne^2}{\epsilon_0 \, m_e \, \omega^2 \, dl} = \frac{e^2}{8\pi^2 \, c \, \epsilon_0 \, m_e} \, \frac{1}{f} \int n \, dl
$$

or in terms of *differential* carrier phase advance,

$$
\Delta\phi_{2-1} = \Delta\phi_2 - \Delta\phi_1 = 1.34 \times 10^{-7} \left(\frac{1}{f_2} - \frac{1}{f_1}\right) \times \text{TEC} \quad \text{(cycles)}
$$

For a typical ionosphere, TEC=$[1.8, 6.5] \times 10^{17}$ m^{-2}, which gives

$$
\Delta\phi_{2-1} = 4 - 15 \;\; \text{cycles}
$$

Clearly, the number of cycles is only unique up to integer multiples of cycle, so with a a carrier phase advance this great, one cannot hope to measure absolute TEC. The method can still be used for precise measurements of changes in TEC.

[4] if by Faraday rotation one means the angle θ in the previous calculation (4). There is, of course, still a phase shift, but here the phase shift shall not be called a "Faraday effect".

4.2 Group Delay

The final consideration is perhaps the most obvious one: since the propagation velocity $v = c/n$ differs from that of vacuum, there is an ionospheric *group delay* Δt. (See fig. 4)

$$\Delta t = \frac{L}{c} - \int \frac{dl}{c/n} = \frac{1}{c} \int (1-n)dl$$

Using the expression found for $\Delta\phi$, it is a simple matter to calculate the *dual-frequency group delay*:

$$\Delta t_{2-1} = \Delta t_2 - \Delta t_1 = 1.34 \times 10^{-7} \left(\frac{1}{f_2^2} - \frac{1}{f_1^2} \right) \times \text{TEC}.$$

The GPS satellites send a data frame every thirty seconds [12], and the satellite has a 10^{-13} s precision atomic clock, so if one waits for a certain starting time, one can infer the delay from the expected time in this signal.

To calculate the TEC from the measured GPS dual-frequency group delay, we substitute for f_1 and f_2 and arrive at the final result:

$$\text{TEC} = 2.85 \times 10^{25} \times \Delta t_{2-1} \quad \text{electrons/m}^2$$

or, if the delay is measured in nanoseconds and the TEC in "TEC units" ($1 \cdot 10^{16}$ electrons/m^2),

$$\text{TEC}\,(10^{16}\,\text{electrons/m}^2) = 2.85 \times \Delta t_{2-1}(\text{ns}) \tag{5}$$

The same corrections for oblique incidence hold for the TEC calculated from group delay as for Faraday rotation from TEC. (Both relations (4) and (5) are linear).

4.3 Changes in TEC

The reason for the wide ranges of values given for TEC in previous sections is that TEC both varies somewhat unpredictably from day to day, and also because TEC varies, more predictably, with time of day (Fig. 5) and the solar activity. One sees at a glance from comparing figs. 2 and 3 that the TEC almost quadruples from a low point to a high point in the sunspot cycle.

Additional nuisances like magnetic storms and ionospheric *scintillation* (irregularities in diffraction and refraction) conspire to make prediction of TEC a full-time job [4].

5 Conclusion

The 1845 discovery of Faraday rotation is a historical example of the diligence and insight of our physicist "ancestors", and at the same time, a story of generality of physical principles; how a discovery in 19th-century lead glass optics found modern-day application in space radio engineering.

I have also found it interesting in its own right to read about GPS, an astounding symbiosis of physics and engineering. That one can measure the total electron content of the ionosphere by this simple yet ingenious technique is impressive, to say the least.

References

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